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Solutions of Klein - Gordon Equation, Using Finite Fourier Cosine Transform

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Abstract: **In this paper the finite Fourier cosine transform is presented to obtain solutions for Klein-Gordon equations. The finite Fourier cosine transform method was successfully applied to Klein - Gordon equation. The initial-boundary value problems for the Klein-Gordon equations are solved on the half range. Such problems posed on time-depend domain. The results reveal that the finite Fourier cosine transform is very effective, simple, convenient and flexible.**

Keywords: **Differential equations, Dispersive, Perturbation, Decomposition.**

1. INTRODUCTION

In this paper we consider two important equations of mathematical physics, the homogeneous Klein-Gordon equation [11]

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \qquad (1)$$

and the non-homogeneous Klein-Gordon_equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \qquad (2)$$

Which appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, and applied physical sciences[14]. Several techniques including finite difference, finite element, scattering, decomposition and variation iteration using Adomian's_polynomials have been used to handle such equations [2,3,11,12]. He [5,14] developed the homotopy perturbation technique for solving such physical problems. In recent years, many research workers have been paid attention, to study the solutions of partial differential equations by using various methods. Among these are the Adonian decomposition method (ADM) [4], He's semi-inverse method [4], the tanh method, the homotopy perturbation method (HPM), the differential transform method and the variational iteration method (VIP) [5,8]. He [7,8] developed the homotopy_perturbation method (HPM) by merging the standard homotopy_and perturbation for solving various physical problems. Various ways have been proposed recently to deal with these partial differential equations, such as Adomian decomposition method. In this work we apply the finite Fourier sine transform method to solve homogeneous and non-homogeneous linear Klein-Gordon_equations [13].

2. THE FINITE FOURIER SINE TRANSFORM

Definition (1):

The finite Fourier cosine transform of a function $u(x,t)$

is defined by[9]:

$$\mathcal{F}_c\big(u(x,t)\big) = \int_0^a u(x,t)\cos\left(\frac{n\pi x}{a}\right)dx \qquad (3)$$

where n is an integer. The function $u(x,t)$ is then called the inverse finite Fourier cosine transform and is given by:

$$u(x,t) = \frac{2}{a}\sum_{n=1}^{\infty} \mathcal{F}_c\big(u(x,t)\big)\cos\left(\frac{n\pi x}{a}\right) \qquad (4)$$

Definition (2):

If u is some function of x and t, then finite Fourier cosine of $\frac{\eth^2 u}{\partial x^2}$ for $0 < x < a$ and $t > 0$ is given by [10]:

$$\mathcal{F}_c\left(\frac{\eth^2 u(x,t)}{\partial x^2}\right) = \frac{-n^2\pi^2}{a^2}\mathcal{F}_c(u(x,t)) - \ [u_x(0,t) - u_x(a,t)\cos(n\pi)] \quad (5)$$

where u_x denotes the partial derivative with respect to x.

To illustrate the basic idea of this method, we consider a general non-homogeneous linear partial differential equation of the form:

$$Au_{tt}(x,t) + Bu_{xx}(x,t) + Cu(x,t) = h(x,t) \qquad (6)$$

with boundary conditions:

$$u(x,t) = u_x(0,t) = u_x(a,t) = 0 \qquad (7)$$

and initial conditions:

$$u(x,0) = f(x) \qquad (8)$$

$$u_t(x,0) = g(x) \qquad (9)$$

where $A, B,$and C are constants. Taking the finite Fourier cosine transform of both sides of Eq(6), we obtain

$$A\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t)) + \left(C - B\frac{n^2\pi^2}{a^2}\right)\mathcal{F}_c(u(x,t)) + \frac{n\pi}{a}[u_x(0,t) - u_x(a,t)\cos(n\pi)] = \mathcal{F}_c\,(h(x,t)) \qquad (10)$$

using the boundary condition (7) and associating like terms, Eq(10)becomes

$$A\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t)) + \left(C - B\frac{n^2\pi^2}{a^2}\right)\mathcal{F}_c(u(x,t)) = \mathcal{F}_c\,(h(x,t)) \qquad (11)$$

which is a second order ordinary differential equation, and has the following solutions:

Case1: if $C > B\frac{n^2\pi^2}{a^2}$ then, solution of Eq(11)is:

$$\mathcal{F}_c(u(x,t)) = c_1\cos(\beta_0 t) + c_2\sin(\beta_0 t) + \rho_n(t) \qquad (12)$$

Case2: if $C < B\frac{n^2\pi^2}{a^2}$ then, solution of Eq(11) is:

$$\mathcal{F}_c(u(x,t)) = c_1\cosh(\beta_0 t) + c_2\sin h(\beta_0 t) + \rho_n(t) \qquad (13)$$

where $\beta_0 = \sqrt{\frac{Ca^2 - Bn^2\pi^2}{Aa^2}}$ and $\rho_n(t)$ is the particular solution of Eq(11).

It is easy to show that $c_1 = \rho_n(0) + f(x)$ and $c_2 = \frac{\rho_n(0)+g(x)}{\beta_n}$ by applying the finite Fourier transform to the initial conditions (8) and (9).

Case3: if $C = B\frac{n^2\pi^2}{a^2}$, then the solution of Eq(11) is:

$$\mathcal{F}_c(u(x,t)) = f(x) + tg(x)\frac{1}{A}\int_0^t (t-\tau)\mathcal{F}_c\,(h(x,t))\,dt \qquad (14)$$

Taking the inverse finite Fourier sine transform to get the final solution using Eq(4).

3. APPLICATIONS

The finite Fourier transforms are used to solve differential equations arising in boundary value problems of physics and mechanics [9]. In this section we will apply the finite Fourier sine transform to solve homogeneous and non-homogeneous linear Klein-Gordon equations [12]:

3.1 Example:

Consider the following boundary value problem [9]

$$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 t}{\partial x^2} \qquad (15)$$

with boundary conditions:

$$u(0,t) = u_x(0,t) = u_x(3,t) = 0\ , \qquad (16)$$

and initial conditions:

$$u(x,0) = 0\,,\ \ 0 < x < 3 \qquad (17)$$

$$u_t(x,t) = 12\cos(\pi x) + 16\cos(3\pi x)\ - 8\cos(5\pi x) \qquad (18)$$

Taking finite Fourier cosine transform of both sides of Eq(15), and using Eq(5) (with $a = 3$), we obtain

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t)) = 16\left[-\frac{n^2\pi^2}{9}\mathcal{F}_c(u(x,t)) - [u_x(0,t) - u_x(3,t)\cos(n\pi)]\right] \quad (19)$$

then, using the boundary conditions (16), Eq(18) becomes

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t)) = -\frac{16}{9}n^2\pi^2\mathcal{F}_c(u(x,t)) \qquad (20)$$

which is a second order ordinary differential equation, and has a solution

$$\mathcal{F}_c(u(x,t)) = A\cos\left(\frac{4n\pi}{3}\,t\right) + B\sin\left(\frac{4n\pi}{3}\,t\right) \qquad (21)$$

where A and B are arbitrary constants of integration.

Using conditions (17) and Eq(21), we have

$$\mathcal{F}_c(u(x,t)) = A + 0 = 0\ \rightarrow A = 0 \qquad (22)$$

and using condition (18) with Eq(21), we obtain:

$$\mathcal{F}_c(u(x,t)) = -\frac{3n\pi}{4}\,A.0 + \frac{3n\pi}{4}\,B.1 = \mathcal{F}_c(12\cos(\pi x) + 16\cos(3\pi x)\ - 8\cos(5\pi x)\,) \ (23)$$

Hence

$$B = \begin{cases} \frac{9}{2\pi} & ,for\ n = 3 \\ \frac{2}{\pi} & ,\ for\ n = 9 \\ \frac{-3}{5\pi} & ,\ for\ n = 15 \\ 0\,, & for\ all\ other\ values\ of\ n \end{cases} \qquad (24)$$

Therefore

$$\mathcal{F}_c(u(x,t)) = \begin{cases} \frac{9}{2\pi}\sin(4\pi t) & ,for\ n = 3 \\ \frac{2}{\pi}\sin(12\pi t) & ,\ for\ n = 9 \\ \frac{-3}{5\pi}\sin(20\pi t) & ,\ for\ n = 15 \\ 0\,, & for\ all\ other\ values\ of\ n \end{cases} \qquad (25)$$

Taking inverse Fourier sine transform of Eq(23), we obtain

$$u(x,t) = \frac{3}{\pi}\sin(4\pi t)\cos(\pi x) + \frac{4}{3\pi}\sin(12\pi t)\cos(3\pi x) + \frac{-2}{5\pi}\sin(20\pi t) \quad (26)$$

which is the required solution.

3.2 The homogeneous linear Klein-Gordon equation:

We next investigate the Klein-Gordon equation[1,13]:

$$u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 0 \qquad (27)$$

with boundary conditions:

$$u(o,t) = u_x(0,t) = u_x(a,t) = 0 \qquad (28)$$

and initial condition:

$$u(x,0) = 0 \qquad (29)$$

$$u_t(x,0) = x \qquad (30)$$

Taking finite Fourier sine transform of both sides of Eq(25), and using Eq(5), we obtain

$$\mathcal{F}_c(u_{tt}(x,t)) = \frac{-n^2\pi^2}{a^2}\mathcal{F}_c(u_{tt}(x,t)) + [u_x(0,t) - u_x(a,t)\cos(n\pi)] + \mathcal{F}_c(u_{tt}(x,t))(n,t) = 0 \qquad (31)$$

Then using the boundary conditions (28) and associating like terms, we get

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t)) + \left(\frac{n^2\pi^2}{a^2} + 1\right)\mathcal{F}_c(u(x,t)) = 0 \qquad (32)$$

or

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t)) + \left(\frac{n^2\pi^2 + a^2}{a^2}\right)\mathcal{F}_c(u(x,t)) = 0 \qquad (33)$$

which is a second order homogeneous ordinary differential equations, and has a solution:

$$\mathcal{F}_c(u(x,t)) = c_1\cos(\beta_n t) + c_2\sin(\beta_n t) \qquad (34)$$

where

$$\beta_n = \frac{\sqrt{n^2\pi^2 + a^2}}{a} \qquad (35)$$

c_1 and c_2 are arbitrary constants of integration, taking finite Fourier sine transform of the initial conditions (29), we get

$$c_1 = 0 \qquad (36)$$

then Eq(33) becomes:

$$\mathcal{F}_c(u(x,t)) = c_2\sin(\beta_n t) \qquad (37)$$

Using initial condition (30), we obtain

$$c_2 = \frac{a^2}{n^2\pi^2\beta_n}(\cos(n\pi) - 1) \qquad (38)$$

then Eq(37) becomes

$$\mathcal{F}_c(u(x,t)) = \frac{a^2}{n^2\pi^2\beta_n}(\cos(n\pi) - 1)\sin(\beta_n t) \qquad (39)$$

Taking inverse Fourier sine transform of Eq(39), we get

$$u(x,t)=\frac{2}{a}\sum_{n=1}^{\infty}\frac{a^2}{n^2\pi^2\beta_n}(\cos(n\pi)-1)\ \sin(\beta_n t)\sin\left(\frac{n\pi}{a}x\right) \qquad (40)$$

since $u(0,t)=0$ from (25) or

$$u(x,t)=\frac{2a^2}{\pi}\sum_{n=0}^{\infty}\frac{(\cos(n\pi)-1)}{n^2\sqrt{n^2\pi^2+a^2}}\sin\left(\frac{\sqrt{n^2\pi^2+a^2}}{a}t\right)\sin\left(\frac{n\pi}{a}x\right) \qquad (41)$$

which is the required solution.

3.3 The in-homogeneous linear Klein-Gordon equation:

We next consider the in-homogeneous linear Klein-Gordon equation [1,13]:

$$u_{tt}(x,t)-u_{xx}(x,t)+u(x,t)=2\sin x \qquad (42)$$

with boundary conditions:

$$u(0,t)=u_x(0,t)=u_x u(a,t)=0 \qquad (43)$$

and initial conditions:

$$u(x,0)=\sin x \quad , \quad 0<x<a \qquad (44)$$

$$u_t(x,0)=1 \quad , \quad 0<x<a \qquad (45)$$

Taking finite Fourier cosine transform of both sides of Eq(42), and using Eq(5), we obtain

$$\mathcal{F}_c(u_{tt}(x,t))=\frac{-n^2\pi^2}{a^2}\mathcal{F}_c(u(x,t))+\frac{n\pi}{a}\,[u(0,t)-u(a,t)\cos(n\pi)]+\mathcal{F}_c(u(x,t))=\mathcal{F}_c(2\sin x\,) \qquad (46)$$

Then using the boundary conditions (43) and associating like terms, we get

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t))+\left(\frac{n^2\pi^2+a^2}{a^2}\right)\mathcal{F}_c(u(x,t))=\ \mathcal{F}_c(2\sin x) \qquad (47)$$

the right hand side can be calculated as follows:

$$\mathcal{F}_c(2\sin x)=\int_0^a 2\sin x\cos\left(\frac{n\pi}{a}x\right)dx$$

and this gives

$$\mathcal{F}_c(2\sin x)=\frac{2a^2}{n^2\pi^2-a^2}(\cos(n\pi)\cos a-1)\,,\ \ n^2\pi^2\neq a^2 \qquad (48)$$

Hence Eq(43) becomes

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t))+\left(\frac{n^2\pi^2+a^2}{a^2}\right)\mathcal{F}_c(u(x,t))=\frac{2a^2}{n^2\pi^2-a^2}(\cos(n\pi)\cos a-1) \quad (49)$$

which is a second order non-homogeneous ordinary differential equations, letting

$$\xi_0=\frac{2a^2}{n^2\pi^2-a^2}\ ,\ \ n^2\pi^2\neq a^2 \qquad (50)$$

Assuming

$$\mathcal{F}_c(u(x,t))=\rho_n=coonstant \qquad (51)$$

Then

$$\frac{d^2}{dt^2}\mathcal{F}_c(u(x,t))=0 \qquad (52)$$

So from Eq(48), we have

$$\left(\frac{n^2\pi^2+a^2}{a^2}\right)\rho_n=\frac{2a^2}{n^2\pi^2-a^2}(\cos(n\pi)\cos a-1) \qquad (53)$$

or

$$\rho_n=\frac{2a^4}{n^4\pi^4-a^4}(\cos(n\pi)\cos a-1) \qquad (54)$$

Hence the general solution is:

$$\mathcal{F}_c(u(x,t))=c_1\cos(\beta_n t)+c_2\sin(\beta_n t)+\rho_n \qquad (55)$$

where β_n is as defined in Eq(35), and

$$\rho_n=\frac{\xi_0}{\beta_n^2}(\cos(n\pi)\sin(a)-1) \qquad (56)$$

Upon using the initial condition (43) gives

$$c_1=\frac{1}{\beta_n}(\cos(n\pi)-1) \qquad (57)$$

Using condition (42) with (3), we have

$$\mathcal{F}_c(u(x,t))=\beta_n c_2=\mathcal{F}_c(1)=\int_0^a\cos\left(\frac{n\pi}{a}x\right)dx=\frac{a}{n\pi}\sin\left(\frac{n\pi}{a}x\right)\Big|_0^a=0 \quad (58)$$

Hence $c_2=0$ (59)

Eq(52) becomes

$$\mathcal{F}_c(u(x,t))=c_1\cos\beta_n+\rho_n \qquad (60)$$

Taking inverse finite Fourier cosine transform, we obtain:

$$u(x,t)=\frac{2}{a}\sum_{n=1}^{\infty}(c_n\beta_n+\rho_n)\cos\left(\frac{n\pi}{a}x\right) \qquad (61)$$

Substituting c_1 , β_n , ρ_n in (), we get

$$u(x,t)=2a\sum_{n=1}^{\infty}\frac{\cos(a)\cos(n\pi)}{n^2\pi^2+a^2}\left(\frac{2a^2}{n^2\pi^2-a^2}+\cos\left(\frac{\sqrt{n^2\pi^2+a^2}}{a}t\right)\right)\cos\left(\frac{n\pi}{a}x\right) \quad (62)$$

which is the required solution.

4. CONCLUSION

After the direct application of finite Fourier cosine transform method and from the results obtained, we can say that this method is easy to implement and effective. As a result, the conclusion that comes through this work, is that the finite Fourier cosine transform method can be applied to other partial differential equation, due to the efficiency and capability in the application to get the possible results.

REFERENCES

[1] Abdallah.H.A, Solutions of Klein - Gordan equations, using Finite Fourier Sine Transform IOSR Journal of Mathematics (IOSR-JM), 2017, Volume 13, Issue6Ver. IV (Nov. - Dec. 2017), PP 19-*24.*

[2] S. Abbasbandy. Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Internet.

[3] A.M. Wazwaz. The modified decomposition method for analytic treatment of differential equations. Appl. Math. Comput. 2006, pp. 165-176.

[4] G. Adomian. Solving Frontier Problems of Physics: The Decomposition Method. Boston: Kluwer Acad. Publ.1994.

[5] J.H. He. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering. 1999.

[6] J.H. He. Variational iteration method a kind of nonlinear analytical technique: some examples. International Journal of Nonlinear Mechanics. 1999.

[7] J.H. He. Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and Computation. 2003.

[8] E. Hesameddini, H. Latifizadeh. A new vision of the He's homotopy perturbation method. International Journal of Nonlinear Sciences and Numerical Simulation, 2009.

[9] E. Hesameddini, H. Latifizadeh. Reconstruction of variational iteration algorithms using the Laplace transform. International Journal of Nonlinear Sciences and Numerical Simulation. 2009.

[10] S.L.Kakani, C.Hemajani, Mathematical physics,CBS Publisher and Distributors, New Dalhi (India),2004.

[11] S.T. Mohyud-Din, M.A. Noor, K.I. Noor. some relatively new techniques for nonlinear problems. Math. Porb.Eng. Article ID 234849, 25 pages, doi: 10.1155/2009/234849, 2009.

[12] S.T. Mohyud-Din, A. Yildirim. Variational iteration method for solving Klein- Gordon equations. Journal of Applied Mathematics, Statistics and Informatics, 2010.

[13] Jagdev Singh, Devendra Kumar and Sushila Rathore, Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinea Klein- Gordon Equations, Journal of Information and Computing Science, ISS 1746-765 England, UK,Vol. 7, No. 2, 2012, pp. 131-139.

[14] A. Yildirim. An Algorithm for Solving the Fractional Nonlinear Schröndinger Equation by Means of the Homotopy Perturbation Method. International Journal of Nonlinear Science and Numerical Simulation. 2009.